An introduction to Einstein constraints and the seed-to-solution method

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Outline of this talk

1. Topic for this lecture
2. Exact vs. asymptotic localization problems
3. A parametrization of localized initial data sets
4. Localization in cone-like domains
5. Main statement with super-harmonic control
6. Concluding remarks

Main collaborators

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1. TOPIC for THIS LECTURE

Einstein's constraint equations

"prescribed curvature problem"

- manifold $(\mathbf{M}, g, k)$ with finitely many asymptotic ends
- unknowns: Riemannian metric g and symmetric $(0,2)$-tensor field k

  extrinsic curvature in the dynamical picture

- matter content: scalar field $H_\star : \mathbf{M} \to \mathbb{R}_+$ vector field $M_\star$
- Einstein's **Hamiltonian and momentum constraints**

$$R_g + (\mathrm{Tr}_g k)^2 - |k|_g^2 = H_\star \qquad \mathrm{Div}_g\big(k - (\mathrm{Tr}_g k)g\big) = M_\star$$

Notation

It is convenient to introduce the $(2,0)$-tensor h by $h := \big(k - \mathrm{Tr}_g(k)g\big)^{\sharp\sharp}$

$$\mathcal{H}(g,h) := R_g + \frac{1}{2}(\mathrm{Tr}_g h)^2 - |h|_g^2 \qquad \mathcal{M}(g,h) := \mathrm{Div}_g h$$

$$\mathcal{G}(g,h) := \big(\mathcal{H}, \mathcal{M}\big)(g,h) = (H_\star, M_\star)$$

In the dynamical picture, $\mathcal{G}(g,h)$ is a spacetime vector.

Vast and rich literature

- Conformal method Lichnerowicz (1960s), Choquet-Bruhat, Chrusciel, Corvino, Delay, Dilts, Galloway, Gicquaud, Holst, Isenberg, Maxwell, Mazzeo, Miao, Pollack, Schoen, etc.
- Variational method Corvino, Corvino-Schoen, Chrusciel-Delay, Carlotto-Schoen, etc.

Major achievements

- existence of initial data, explicit constructions, physically relevant solutions
- general relativity, Riemannian geometry

- numerous classes of solutions: compact, various types of asymptotic ends
- including gluing techniques, combine two different solutions together

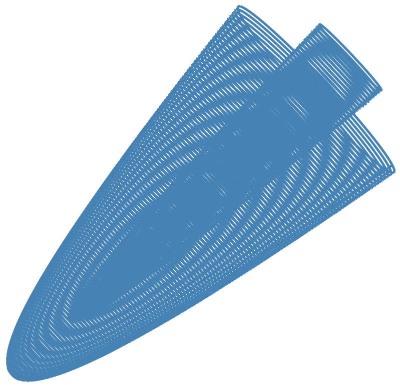
- A. Carlotto, *The general relativistic constraint equations,* Living Reviews in Relativity (2021).
- Very active subject: Lee, Lesourd, Corvino, Pasqualotto, etc.

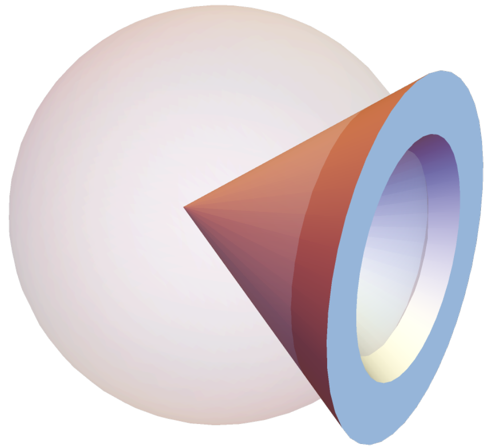
Shielding gravity: localization at infinity

- asymptotically Euclidean initial data sets
- phenomena of anti-gravity (or shielding) — Carlotto and Schoen; Chruściel and Delay
- solutions that are localized at infinity
  - The Positive Mass Theorem implies restrictions on gluing at infinity.
  - identically Euclidian near infinity except in a cone with possibly arbitrary small angle
- Other recent developments
  - S. Aretakis, S. Czimek, I. Rodnianski: characteristic gluing problem
  - Y.-C. Mao and Z.-K. Tao: localization "a la Carlotto-Schoen" in narrow domains

Localization with (super-)harmonic control

- Carlotto and Schoen
  - solutions with *sub-harmonic* control — r^p with $p \in \left(\frac{n-2}{2}, n-2\right)$
  - conjecture: gluing should be possible at harmonic level
- Localization results with harmonic and super-harmonic control
  - PLF & The-Cang Nguyen, 2020: *The seed-to-solution method for the Einstein constraint equations*
  - Bruno Le Floch & PLF, 2023





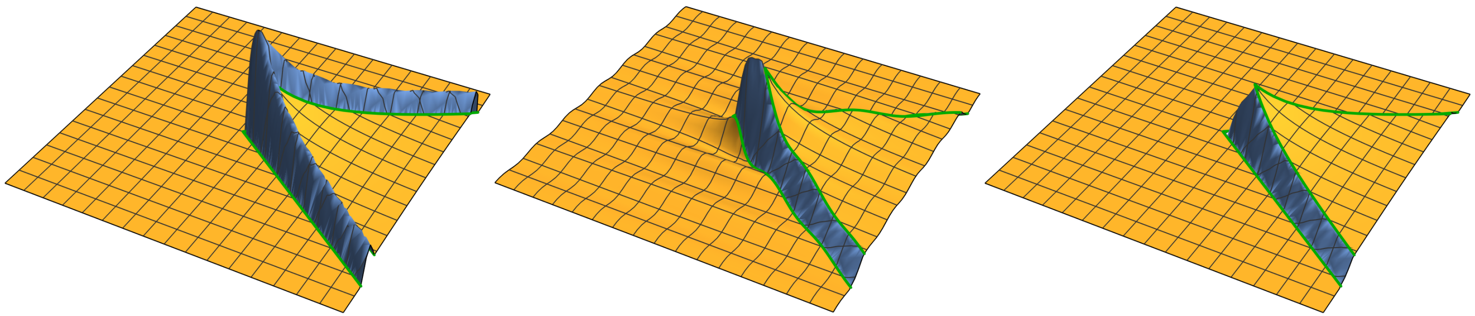


Figure: Gluing of the Euclidean metric and the Schwarzschild metric.
Left: *exact localization with sub-harmonic control*
Middle: *asymptotic localization with harmonic control*
Right: *exact localization with harmonic control*

2. EXACT vs. ASYMPTOTIC LOCALIZATION PROBLEMS

Theorem. The seed-to-solution method **(PLF–Nguyen)**

Given any seed data set $(\mathbf{M}, g_1, h_1)$ on a 3-manifold (with a single end, say): a Riemannian metric g_1 and a symmetric two-tensor h_1 satisfying (suitable smallness conditions and) $1/2 < p_G \leqslant \min(1, p_M)$, $1/2 < p_M < +\infty$

$$g_1 = g_{\mathrm{Eucl}} + \mathcal{O}(r^{-p_G}) \qquad h_1 = \mathcal{O}(r^{-p_G-1})$$

$$\mathcal{H}(g_1, h_1) = \mathcal{O}(r^{-p_M-2}) \qquad \mathcal{M}(g_1, h_1) = \mathcal{O}(r^{-p_M-2})$$

there exists a solution (g, h) to the vacuum Einstein equations $\mathcal{G}(g, h) = 0$.

- **Sub-harmonic decay:** $p_M < 1$

$$g = g_1 + \mathcal{O}(r^{-p_M}) \qquad h = h_1 + \mathcal{O}(r^{-p_M-1})$$

- **Harmonic decay:** $p_M = 1$ $\qquad \mathcal{H}(g_1, h_1)$ and $\mathcal{M}(g_1, h_1)$ in $L^1(M)$

$$g = g_1 + \frac{\widetilde{m}}{r} + o(r^{-1}) \qquad h = h_1 + \mathcal{O}(r^{-2})$$

- **Super-harmonic decay:** $p_M > 1$ $\qquad p = \min(p_G + 1, p_M, 2)$

$$g = g_1 + \frac{\widetilde{m}}{r} + \mathcal{O}(r^{-p}) \qquad h = h_1 + \mathcal{O}(r^{-2})$$

Mass modulator $\widetilde{m} = \widetilde{m}(g_1, h_1) = \mathrm{const.} \int_{\mathbf{M}} \mathcal{H}(g_1, h_1)\, dV_{g_1} + \mathcal{O}(\mathcal{G}(g_1, h_1)^2)$

Exact localization problem

– Vacuum constraint Einstein equations

Decompose asymptotic infinity into three angular regions

- $\mathscr{C}_a$: cone with (possibly arbitrarily small) angle $a \in (0, 2\pi)$
- $\mathscr{C}^c_{a+\varepsilon}$: complement of the same cone with (slightly) larger angle $a + \varepsilon$
- $\mathcal{T}^\varepsilon_a$: remaining transition region

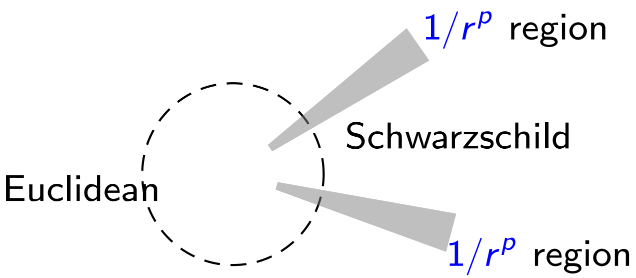
$\mathscr{C}_a$ and $\mathscr{C}^c_{a+\varepsilon}$: the metric coincides with Euclidean/Schwarzschild ones

solve the vacuum Einstein equations in the transition region $\mathscr{T}_{a,\varepsilon}$

– *Sub-harmonic control in* $\mathscr{T}_{a,\varepsilon}$*, that is,* r^{-p} *with* $p \in (1/2, 1)$

– Question raised by Carlotto and Schoen

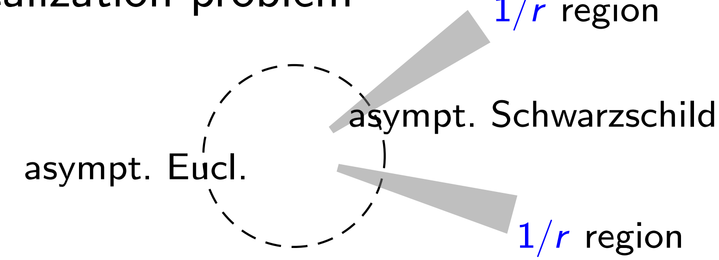
construct solutions (with prescribed asymptotic)
enjoying the $1/r$ harmonic decay in all angular directions



Asymptotic localization problem

slightly *relax the localization condition*

- asymptotic at a super-harmonic rate to prescribed metrics
- physically as natural as the exact localization problem



Theorem. The asymptotic localization problem (PLF–Nguyen)

- Vacuum Einstein equations on a manifold **M** with a single asymptotic end
- Decompose asymptotic infinity into three asymptotic angular regions

$$\mathscr{C}_a \cup \mathscr{C}^c_{a+\varepsilon} \cup \mathscr{T}_{a,\varepsilon} \subset \mathbb{R}^3$$

By considering (for instance) the Euclidean metric g_{Eucl} and the Schwarzschild metric $g_{\text{Sch}} = (1 + 2m_{\text{Sch}}/r)\, g_{\text{Eucl}}$ (with mass $m_{\text{Sch}} > 0$),
there exists a solution to the vacuum Einstein equations $\mathcal{G}(g, h) = 0$:

$$g = g_{\text{Eucl}} + \mathcal{O}(r^{-q}) \qquad \text{in } \mathscr{C}^c_{a+\varepsilon}$$

$$g = g_{\text{Sch}} + \mathcal{O}(r^{-q}) \qquad \text{in } \mathscr{C}_a \qquad q \in (1, 2)$$

$$g = g_{\text{Eucl}} + \mathcal{O}(r^{-1}) \qquad \text{in } \mathscr{T}_{a,\varepsilon}$$

3. A PARAMETRIZATION of INITIAL DATA SETS

Definition

A **localized reference set** $(\mathbf{M}, g_0, h_0, \omega)$ consists of
– Riemannian manifold $(\mathbf{M}, g_0)$
– symmetric $(2,0)$-tensor h_0 on $\mathbf{M}$
– **localization weight** $\omega \in [0, +\infty]$

- a smooth open set $\Omega \subset \mathbf{M}$, referred to as the **gluing domain**

$$\omega \begin{cases} > 0 & \text{in the gluing domain } \Omega \\ \to 0 & \text{when approaching } \partial\Omega \text{ from } \Omega \\ \to 0 & \text{at asymptotic ends} \\ = +\infty & \text{in } {}^{\complement}\Omega \end{cases}$$

- $\omega := \frac{\mathbf{b}}{\mathbf{r}} \in [0, +\infty]$
  - *asymptotic radius* $\mathbf{r} : \mathbf{M} \to [1, +\infty)$ $\qquad$ $\mathbf{r} \simeq \left(1 + (\mathbf{d}_{g_0}(\cdot, y_0))^2\right)^{1/2}$ for some (fixed) point $y_0 \in \mathbf{M}$
  - *boundary distance* $\mathbf{b} : \mathbf{M} \to (0, 1]$ $\qquad$ large integer $N \geqslant 1$

$$\mathbf{b} \begin{cases} \simeq \min\left(1, (\mathbf{d}_{\partial\Omega}/\mathbf{r})^N\right) & \text{in the gluing domain } \Omega \\ = +\infty & \text{in the fixed domain } {}^{\complement}\Omega \end{cases}$$

- *Weighted Hölder spaces* $C_p^{l,\alpha}(\mathbf{M}, g_0, \omega)$ with

$$\|f\|_{g_0,\omega,p}^{l,\alpha} = \sum_{|L|\leqslant l} \sup_{\mathbf{M}} \left(\mathbf{b}\,\omega^{p+|L|}|\nabla^L f|_{g_0}\right) + \sum_{|L|=l} \sup_{\mathbf{M}} \left(\mathbf{b}\,\omega^{p+|L|+\alpha}\,[\![\nabla^L f]\!]_{g_0,\alpha}\right)$$

- *Weighted Lebesgue spaces* (and Sobolev spaces, etc.):

$$\|f\|_{L_p^2(\mathbf{M},g_0,\omega)}^2 := \int_{\mathbf{M}} |f|_{g_0}^2\,\mathbf{b}\,\omega^{2p-n}\,dV_{g_0}$$

- *Weighted Lebesgue-Hölder spaces* $L^2C_\omega^{l,\alpha}(\mathbf{M}, g_0)$ with

$$\left(|\!|\!| f |\!|\!|_{g_0,\omega,p}^{l,\alpha}\right)^2 := \left(\|f\|_{L_p^2(\mathbf{M},g_0,\omega)}\right)^2 + \left(\|f\|_{C_p^{l,\alpha}(\mathbf{M},g_0,\omega)}\right)^2$$

Definition

Consider a localized reference set $(\mathbf{M}, g_0, h_0, \omega)$ and some parameters $\varepsilon_G, \varepsilon_M > 0$. Consider **decay** and **accuracy exponents** (p_G, q_G) and (p_M, q_M)

$$(p_M, q_M) \geqslant (p_G, q_G)$$

A (p_G, q_G, p_M, q_M)-**localized seed data set** $(g_1, h_1, H_\star, M_\star)$ consists of:

- **Near-reference data**

  g_1 is a Riemannian metric

  h_1 is a symmetric $(2,0)$-tensor

  $$\|g_1 - g_0\|^{N,\alpha}_{g_0,\omega,p_G} \leqslant \varepsilon_G \qquad \|h_1 - h_0\|^{N-1,\alpha}_{g_0,\omega,q_G} \leqslant \varepsilon_G$$

- **Near-Einsteinian data**

  $H_\star$ is a scalar field and $M_\star$ is a vector field and

  $$\left\|\mathcal{H}(g_1, h_1) - H_\star\right\|^{N-4,\alpha}_{g_0,\omega,p_M+2} \leqslant \varepsilon_M$$

  $$\left\|\mathcal{M}(g_1, h_1) - M_\star\right\|^{N-3,\alpha}_{g_0,\omega,q_M+1} \leqslant \varepsilon_M$$

- metrics with possibly slow decay
- treat, together, the Einstein operator on (g_1, h_1) and the matter content

We seek to make a "projection" of each (g_1, h_1) on the "solution manifold".

Variational framework

Given (f, V) consider the functional

$$\int_{\mathbf{M}} \Big(\frac{1}{2} \big| d\mathcal{H}^*_{(g_1,h_1)}[u, Z] \big|^2_{\mathbf{b}} \, \omega^{2p-n} + \frac{1}{2} \big| d\mathcal{M}^*_{(g_1,h_1)}[u, Z] \big|^2_{\mathbf{b}} \, \omega^{2q-n} - f\, u - g_1(V, Z) \Big) \, dV_{g_1}$$

Euler-Lagrange equation

(u, Z) minimizer: fourth-order PDEs

$$g - g_1 = \mathbf{b}\, \omega^{2p-n} d\mathcal{H}^*_{(g_1,h_1)}[u, Z]$$

$$h - h_1 = \mathbf{b}\, \omega^{2q-n} d\mathcal{M}^*_{(g_1,h_1)}[u, Z]$$

$$(f, V) = d\mathcal{G}_{(g_1,h_1))}[g - g_1, h - h_1]$$

Fixed-point's Picard scheme

- quadratic part of the Einstein operator — decay properties of nonlinearities

$$\mathcal{Q}_{(g_1,h_1)}[g_2, h_2] := \mathcal{G}(g, h) - \mathcal{G}(g_1, h_1) - d\mathcal{G}_{(g_1,h_1)}[g_2, h_2]$$

- study sequences (f_i, V_i) and (g_i, h_i)

$$(f_i, V_i) := -\mathcal{Q}_{(g_1,h_1)}[g_{i-1}, h_{i-1}] + (H_\star, M_\star) - \mathcal{G}(g_1, h_1)$$

$$(g_i, h_i) := (d\mathcal{G})^{-1}_{(g_1,h_1)}(f_i, V_i)$$

Operator $d\mathcal{G}_{(g_0,h_0)} \circ$ **(weights)** $\circ\, d\mathcal{G}^*_{(g_0,h_0)}$

with $w_p = \mathbf{b}\,\omega^{2p-n}$ and $w_q = \mathbf{b}\,\omega^{2q-n}$

$$\begin{aligned} d\mathcal{H}\big[w_p\, d\mathcal{H}^*[u]\big] &= (n-1)\Delta(w_p\,\Delta u) + (\nabla^i\nabla^j w_p - w_p\, R^{ij})(\nabla_i\nabla_j u - R_{ij}u) \\ &\quad - (\Delta w_p - 2w_p\,R)\Delta u \\ &\quad + \Big(2R\,\nabla_i w_p + \frac{3}{2}w_p\,\nabla_i R\Big)\nabla^i u + \frac{1}{2}\Big(\Delta(w_p\,R) + (\Delta w_p)R\Big)u \\ d\mathcal{M}\big[w_q\, d\mathcal{M}^*[Z]\big]^i &= -\frac{1}{2}w_q\,(\nabla_j\nabla^j Z^i + \nabla_j\nabla^i Z^j) - \frac{1}{2}\nabla_j w_q\,(\nabla^j Z^i + \nabla^i Z^j) \end{aligned}$$

In the vicinity of an asymptotically Euclidian end: $e_{ij} = \delta_{ij}$

$$\mathcal{A}_{e,w_p}[u] := (n-1)\Delta_e(w_p\,\Delta_e u) + (\partial_i\partial_j w_p)\partial_i\partial_j u - (\Delta_e w_p)\Delta_e u$$

$$\mathcal{B}_{e,w_q}[Z]^i := -\frac{1}{2}w_q\,(\Delta_e Z^i + \partial_j\partial_i Z^j) - \frac{1}{2}(\partial_j w_q)(\partial_j Z^i + \partial_i Z^j)$$

Solution mapping

order of regularity $N \geqslant 4$ and Hölder exponent $\alpha \in (0,1]$

Definition

- a localized reference set $(\mathbf{M}, g_0, h_0, \omega)$ and parameters $\varepsilon_G, \varepsilon_M > 0$
- pairs (p_G, q_G) and (p_M, q_M) with $(p_M, q_M) \geqslant (p_G, q_G)$
- pair of exponents (p, q)

To any (p_G, q_G, p_M, q_M)-localized seed data set $(g_1, h_1, H_\star, M_\star)$,
the (p,q)-**localized seed-to-solution map**
associates a scalar-valued field u and a vector-valued field Z:

$$g = g_1 + \mathbf{b}\, \omega^{2p-n}\, d\mathcal{H}^*_{(g_1,h_1)}(u,Z) \qquad h = h_1 + \mathbf{b}\, \omega^{2q-n}\, d\mathcal{M}^*_{(g_1,h_1)}(u,Z)$$

obey the Einstein constraints $\mathcal{G}(g,h) = (H_\star, M_\star)$

Lebesgue-Hölder norm

$$\| g - g_1 \|^{N,\alpha}_{g_0,\omega,p} \leqslant \varepsilon_G$$

$$\| h - h_1 \|^{N-1,\alpha}_{g_0,\omega,q} \leqslant \varepsilon_G$$

$$\| u \|^{N,\alpha}_{g_0,\omega,p} \lesssim \| \mathcal{H}(g_1,h_1) - H_\star \|^{N-4,\alpha}_{g_0,\omega,p+2} + \varepsilon_G \| \mathcal{M}(g_1,h_1) - M_\star \|^{N-3,\alpha}_{g_0,g_0,\omega,q+1}$$

$$\| Z \|^{N-1,\alpha}_{g_0,\omega,q} \lesssim \varepsilon_G \| \mathcal{H}(g_1,h_1) - H_\star \|^{N-4,\alpha}_{g_0,\omega,p+2} + \| \mathcal{M}(g_1,h_1) - M_\star \|^{N-3,\alpha}_{g_0,\omega,q+1}$$

Our parametrization

Definition

Equivalence relation is defined between two pairs of tensors (g_1, h_1) and (g, h)

$$g = g_1 + \mathbf{b}\,\omega^{2p-n}\, d\mathcal{H}^*_{(g_0,h_0)}(u, Z), \quad h = h_1 + \mathbf{b}\,\omega^{2q-n}\, d\mathcal{M}^*_{(g_0,h_0)}(u, Z)$$

for some scalar field u and vector field Z $\qquad (g, h) \sim_{(\omega,p,q)} (g_1, h_1)$.

Definition

For any given matter data $H_\star, M_\star$ (possibly vacuum data)
the corresponding solution map $\mathcal{S}_{\omega,p,q}$
sending a (seed data) element (g_1, h_1) to one of its representative (exact solution) in the same class $[(g_1, h_1)]$, namely

$$\mathcal{S}_{\omega,p,q} : (g_1, h_1) \in \mathcal{E}_{\varepsilon_G,\varepsilon_M}(p_G, q_G; p_M, q_M) \mapsto (g, h) \in [(g_1, h_1)]$$

is referred to as the **localized seed-to-solution parametrization** for Einstein's constraint equations in the vicinity of the localized reference set $(\mathbf{M}, g_0, h_0, \omega)$.

- make a specific choice of reference g_0, h_0
- establish weighted Poincaré and weighted Korn inequalities
- conditions required: geometry of the gluing domain, decay/accuracy exponents

Coercivity and elliptic regularity

- linearization $d\mathcal{G}_{(g_1,h_1)}$ of the Einstein operator $\mathcal{G}$ around (g_1, h_1)
- not elliptic unless a gauge choice is made
- restrict the deformation (u, Z) to lie in the image of the dual operator $d\mathcal{G}^*_{(g_1,h_1)}$, up to weights that suitably localize the deformation of interest

- invertibility of the fourth-order operator $d\mathcal{G}_{(g_1,h_1)} \circ (\text{weights}) \circ d\mathcal{G}^*_{(g_1,h_1)}$
- weighted Hölder-Sobolev spaces
- weighted Poincaré inequality, weighted Korn inequality
- elliptic system in the sense of Douglis-Nirenberg, Hölder interior regularity estimates