Convergence of a one-phase Stefan problem with Neumann boundary data to a self-similar profile

Danielle Hilhorst[1], Sabrina Roscani[2], Piotr Rybka[3]

[1] CNRS and University Paris-Saclay, France
[2] University of Rosario, Argentina
[3] University of Warsaw, Poland

FJ-LMI and University of Tokyo

Previous results

Before presenting our results, I would first like to recall previous results, which have been or will soon be published :

C. Bataillon, M. Bouguezzi, D. Hilhorst, F. Lequien, H. Matano, F. Rouillard, J.-F. Scheid, Anodic dissolution model with diffusion-migration transport for simulating localized corrosion, Electrochimica Acta 477 (2024).

Meriem Bouguezzi, Danielle Hilhorst, Yasuhito Miyamoto, Jean-François Scheid, Convergence to a self-similar solution for a one-phase Stefan problem arising in corrosion theory. European J. Appl. Math. 34 (2023), 701-737.

The formation of a pit

We consider a pure iron steel in contact with an aqueous solution of sodium chloride (NaCl). One of the major failure mechanisms in an aggressive aqueous solution is pitting corrosion. It is generally associated to the presence of a special anion, namely the chloride ion. The presence of such an ion leads to the formation of small isolated holes (pits) on the surface of the steel that may reach a considerable depth. The life cycle of a stainless alloy decreases in the presence of corrosion.
In this talk, we focus on a physical model which aims to describe the propagation process of one individual corrosion pit.

The formation of a pit

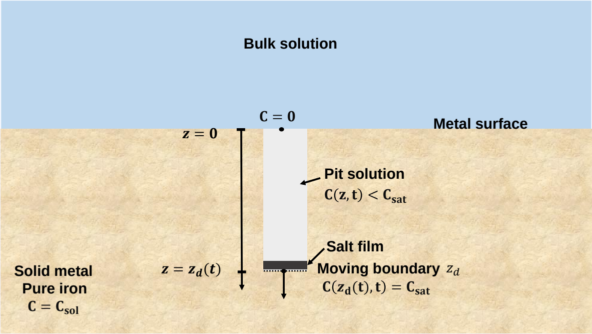


Figure: A one dimensional corrosion pit.

A simplified mathematical problem

The mathematical problem has the form of one-dimensional one-phase Stefan problem, namely

$$
\begin{cases}
u_t = u_{xx}, & t > 0, 0 < x < s(t), \\
u(0,t) = h, & t > 0, \\
u(s(t),t) = 0, & t > 0, \\
\dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \\
s(0) = b_0, & \\
u(x,0) = u_0(x), & 0 < x < b_0
\end{cases}
\tag{1}
$$

where $x = s(t)$ is the unknown free boundary which is to be found together with $u(x,t)$.

The one-dimensional one-phase Stefan problem

Let $h > 0$, $b > 0$. We define the set

$$X^h(b) := \{v \in C[0,\infty),\ v(0) = h,\ v(x) \geqslant 0 \text{ for } 0 \leqslant x < b,$$
$$v(x) = 0 \text{ for } x \geqslant b\} \quad (2)$$

and suppose that

$$u_0 \in X^h(b_0).$$

Let $(u(x,t), s(t))$ be a solution of (1) for all $0 \leqslant t \leqslant T$. We extend u by:

$$u(x,t) = 0 \text{ for } x \geqslant s(t), \quad (3)$$

so that $u(\cdot, t)$ is defined for all $x \geqslant 0$.

Existence and uniqueness

Theorem (Friedman, Schaeffer) Let $h > 0, b > 0$ and $u_0 \in X^h(b)$. Then, there exists a unique classical solution $(u(x,t), s(t))$ of Problem (1) for all $t > 0$. Moreover, the solution (u, s) is such that s is infinitely differentiable on $(0, \infty)$ and u is infinitely differentiable up to the free boundary for all $t > 0$. Furthermore, the function $s(t)$ is strictly increasing in t, and the function u satisfies $0 \leq u \leq h$.

The self-similar solution

We look for a self-similar solution in the form

$$\begin{cases} u(x,t) = U\Big(\dfrac{x}{\sqrt{t+1}}\Big), \\ s(t) = a\sqrt{t+1}, \end{cases} \tag{4}$$

for some positive constant a still to be determined. We set

$$\eta := \frac{x}{\sqrt{t+1}} \tag{5}$$

and deduce that

$$\begin{cases} U_{\eta\eta} + \dfrac{\eta}{2} U_\eta = 0, & 0 < \eta < a, \\ U(0) = h, & U(a) = 0. \end{cases} \tag{6}$$

The unique solution of (6) is given by

$$U(\eta) = h\left[1 - \frac{\int_0^\eta e^{-\frac{s^2}{4}} ds}{\int_0^a e^{-\frac{s^2}{4}} ds}\right] \quad \text{for all } \eta \in (0,a).$$

(7)

Determination of the constant a

It remains to determine the constant a. We write that

$$s'(t) = \frac{a}{2\sqrt{t+1}} = -u_x(s(t),t) = -\frac{U_\eta\left(\frac{s(t)}{\sqrt{t+1}}\right)}{\sqrt{t+1}},$$

which implies that

$$\frac{a}{2} = -U_\eta(a), \tag{8}$$

so that a is characterized as the unique solution of the equation

$$h = \frac{a}{2} e^{\frac{a^2}{4}} \int_0^a e^{-\frac{s^2}{4}} ds. \tag{9}$$

We remark that the function $a = a(h)$ is strictly increasing, which in turn implies that the functional $h \to U$ is strictly increasing.

Change of coordinates

Now, the question is : in what sense can we prove the convergence to the self-similar solution? We are dealing with x and t coordinates as well as with the similarity variable η. This leads us to set $\tau = \ln(t+1)$, and

$$\begin{cases} W(\eta, \tau) = u(x, t), \\ b(\tau) = \dfrac{s(t)}{\sqrt{t+1}}, \end{cases} \tag{10}$$

with η given by (5). We obtain the problem

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0, \tau) = h, \quad W(b(\tau), \tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau), \tau), & \tau > 0, \end{cases} \tag{11}$$

The full problem in self-similar coordinates

Next, we write the full time evolution problem corresponding to the system (11). It is given by

$$\begin{cases} W_\tau = W_{\eta\eta} + \dfrac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\ W(0,\tau) = h, & \tau > 0, \\ W(b(\tau),\tau) = 0, & \tau > 0, \\ \dfrac{db(\tau)}{d\tau} + \dfrac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\ b(0) = b_0, & \\ W(\eta,0) = u_0(\eta), & 0 \leqslant \eta \leqslant b_0. \end{cases} \tag{12}$$

Finally, we note that the stationary solution of Problem (12) coincides with the unique solution of Problem (6), (9) or in other words, with the self-similar solution of Problem (1).

Theorem Let $u_0 \in X^h(b_0) \cap \mathbb{W}^{1,\infty}(0, b_0)$. Let $(W, b) = \left(W(\cdot, \cdot, (u_0, b_0)), b(\cdot, (u_0, b_0))\right)$ be the solution of Problem (17) with the initial data (u_0, b_0). Then

$$\lim_{\tau \to +\infty} W(\eta, \tau) = U(\eta) \quad \text{for all } \eta \in (0, a) \tag{13}$$

and

$$\lim_{\tau \to +\infty} b(\tau) = a \tag{14}$$

where (U, a) is the unique solution of the stationary Problem (6) and (15).

Ideas for the proof

Construct lower and upper solutions and apply a comparison principle.

Main difficulty : Prove that the limit solution satisfies the interface condition

$$\frac{a}{2} = -U_\eta(a). \tag{15}$$

To that purpose, we proved that the derivative W_η converges unformly to its limit as $\tau \to \infty$, using a regularity theorem from the book of Gary Lieberman about parabolic equations.

Criticism of Piotr Rybka : there would not be any way to transpose this proof to the case where the Laplacian in the heat equation would be replaced by a fractional Laplacian.

Today we replace the application of Lieberman's result by passing to the limit in a weak form of the Stefan problem.

The new problem, with a Neumann boundary condition

We study again the same problem, however with the Dirichet boundary condition replaced by a Neumann boundary condition, namely

$$\begin{cases} u_t = u_{xx}, & t > 0, 0 < x < s(t), \\ -u_x(0,t) = h/\sqrt{t+1}, & t > 0, \\ u(s(t),t) = 0, & t > 0, \\ \dfrac{ds(t)}{dt} = -u_x(s(t),t), & t > 0, \\ s(0) = b_0, \\ u(x,0) = u_0(x), & 0 < x < b_0 \end{cases} \tag{16}$$

where $x = s(t)$ is the unknown free boundary which is to be found together with $u(x,t)$. This is another model problem from literature.

The corresponding problem in self-similar coordinates

Next, we write the full time evolution problem corresponding to the system (16). It is given by

$$
\begin{cases}
W_\tau = W_{\eta\eta} + \frac{\eta}{2} W_\eta, & \tau > 0, \quad 0 < \eta < b(\tau), \\
-W_\eta(0,\tau) = h, & \tau > 0, \\
W(b(\tau),\tau) = 0, & \tau > 0, \\
\frac{db(\tau)}{d\tau} + \frac{b(\tau)}{2} = -W_\eta(b(\tau),\tau), & \tau > 0, \\
b(0) = b_0, & \\
W(\eta, 0) = u_0(\eta), & 0 \leqslant \eta \leqslant b_0.
\end{cases}
\tag{17}
$$

The corresponding steady state problem

The associated stationary problem is given by

$$\begin{cases} W_{\eta\eta} + \frac{\eta}{2} W_\eta = 0, \quad 0 < \eta < a, \\ -W_\eta(0) = h, \quad W(a) = 0, \\ \frac{a}{2} = -W_\eta(a). \end{cases} \tag{18}$$

Problem (18) admits a unique solution given by the pair (U, a) such that

$$U(\eta) = h \int_\eta^a e^{-\frac{s^2}{4}}\, ds, \qquad \eta \in [0, a] \tag{19}$$

and a is the unique positive solution of the equation $h = \dfrac{x}{2} e^{\frac{x^2}{4}}$.

Upper- and lower solutions

Definition

We define the linear operator $\mathcal{L}(W) := W_\tau - W_{\eta\eta} - \frac{\eta}{2} W_\eta$. The pair $(\underline{W}, \underline{b})$ is a lower solution of Problem (17) if it satisfies:

$$\begin{cases} \mathcal{L}(\underline{W}) = \underline{W}_\tau - \underline{W}_{\eta\eta} - \frac{\eta}{2}\underline{W}_\eta \leqslant 0, & \tau > 0, \quad 0 < \eta < \underline{b}(\tau), \\ -\underline{W}_\eta(0,\tau) \leqslant h, \quad \underline{W}(\underline{b}(\tau),\tau) = 0, & \tau > 0, \\ \frac{d\underline{b}(\tau)}{d\tau} + \frac{\underline{b}(\tau)}{2} \leqslant -\underline{W}_\eta(\underline{b}(\tau),\tau), & \tau > 0, \\ \underline{b}(0) \leqslant b_0, & \\ \underline{W}(\eta, 0) \leqslant u_0(\eta), & 0 \leqslant \eta \leqslant \underline{b}(0). \end{cases} \quad (20)$$

Similarly, $(\bar{W}, \bar{b})$ is an upper solution of the Problem (17) if it satisfies Problem (20) with all $\leqslant$ replaced with $\geqslant$.

Comparison principle

One can prove the following comparison principle.

Theorem

Let $(W_1(\eta,\tau), b_1(\tau))$ and $(W_2(\eta,\tau), b_2(\tau))$ be respectively lower and upper solutions of (17) corresponding respectively to the data (h_1, u_{01}, b_{01}) and (h_2, u_{02}, b_{02}).
If $b_{01} \leqslant b_{02}$, $h_1 \leqslant h_2$ and $u_{01} \leqslant u_{02}$, then $b_1(\tau) \leqslant b_2(\tau)$ for $\tau \geqslant 0$ and $W_1(\eta,\tau) \leqslant W_2(\eta,\tau)$ for $\eta \geqslant 0$ and $\tau \geqslant 0$.

In what follows we will keep using the notation $W(\eta,\tau,(u_0,b_0))$ and $b(\tau,(u_0,b_0))$ for the solution pair associated with the initial data (u_0,b_0).

Stationary lower and upper solutions

In order to find stationary lower and upper solutions, we suppose that λ is a positive constant and consider the perturbed problem

$$\begin{array}{lll} (i) & W_{\eta\eta}(\eta) + \lambda \dfrac{\eta W_\eta(\eta)}{2} = 0 & 0 < \eta < b_\lambda, \\ (ii) & -W_\eta(0) = \tilde{h}, & W(b_\lambda) = 0, \\ (iii) & \dfrac{b_\lambda}{2} = -W_\eta(b_\lambda). & \end{array} \tag{21}$$

whose solution is given by the pair (W_λ, b_λ), where

$$U_\lambda(\eta) = \tilde{h} \int_\eta^{b_\lambda} e^{-\lambda s^2/4} \mathrm{ds},$$

where b_λ is the unique solution of the equation

$$\tilde{h} = \frac{b_\lambda}{2} e^{\lambda b_\lambda^2/4}. \tag{22}$$

Stationary lower and upper solutions

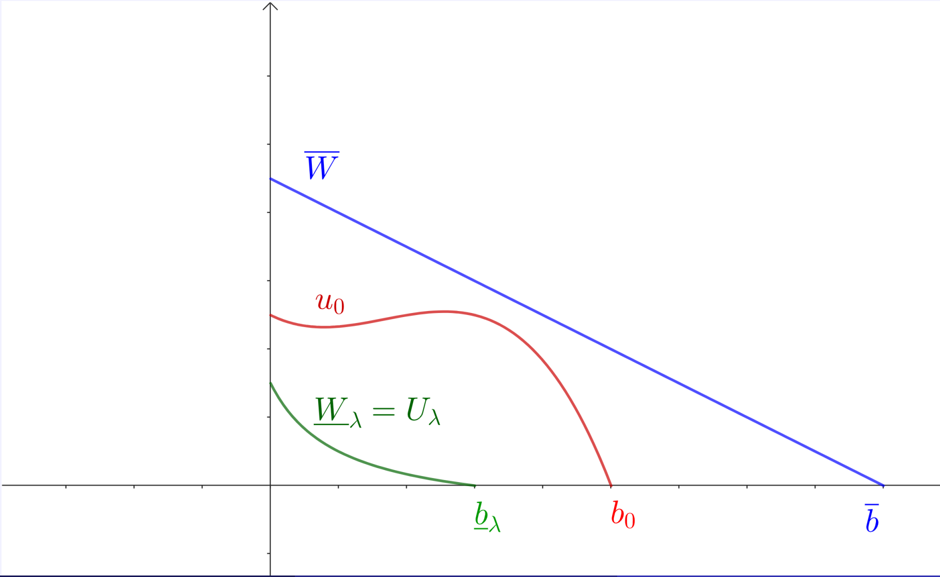
In order to define a lower solution, we suppose that $\tilde{h} \leq h$ and that $\lambda > 1$. Then the solution (W_λ, b_λ) of Problem (21) is a lower solution of Problem (17).

Similarly, set $\lambda = 0$, and suppose that $\overline{b} \geq \max(b_0, 2h)$, and define

$$\bar{W}(\eta) = \frac{\overline{b}}{2}(\overline{b} - \eta), \text{ if } 0 \leq \eta \leq \overline{b}.$$

Then the pair$(\bar{W}, \overline{b})$ is an upper solution of Problem (17).

A typical configuration



First convergence results

Next, we prove the monotonicity in time of the solution pair (W, b) of the time evolution Problem (17) with the two initial conditions $(\bar{W}, \bar{b})$ and (W_λ, b_λ). We recall that $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$ denotes the solution pair of Problem (17) with the initial conditions (u_0, b_0).

Lemma Let $(\bar{W}, \bar{b})$ and $(\underline{W}_\lambda, \underline{b}_\lambda)$ be defined above. The functions $W(\eta, \tau, (\bar{W}, \bar{b}))$ and $b(\tau, (\bar{W}, \bar{b}))$ are nonincreasing in time. Furthermore, there exist a positive constant $\bar{b}_\infty$ and a function $\phi \in L^\infty(0, \bar{b}_\infty)$ such that

$$\lim_{\tau \to +\infty} W(\eta, \tau, (\bar{W}, \bar{b})) = \phi(\eta) \quad \text{for all} \quad \eta \in (0, \bar{b}_\infty), \tag{23}$$

$$\lim_{\tau \to +\infty} b(\tau, (\bar{W}, \bar{b})) = \bar{b}_\infty. \tag{24}$$

Convergence results

The functions $W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda))$ and $b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda))$ are nondecreasing in time. Furthermore, there exist a positive constant $\underline{\mathrm{b}}_\infty$ and a function $\psi \in L^\infty(0, \underline{\mathrm{b}}_\infty)$ such that

$$\lim_{\tau \to +\infty} W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) = \psi(\eta) \quad \text{for all} \quad \eta \in (0, \underline{\mathrm{b}}_\infty), \tag{25}$$

$$\lim_{\tau \to +\infty} b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)) = \underline{\mathrm{b}}_\infty. \tag{26}$$

It remains to prove that

$$\phi = \psi = U$$

and that

$$\underline{b}_\infty = \overline{b}_\infty = a.$$

An extra a priori estimate

Since $(\underline{W}_\lambda, \underline{b}_\lambda)$ and since $(\overline{W}, \overline{b})$ are respectively lower and upper solutions, we have the L^∞ estimates

$$\underline{W}_\lambda \leq W(\tau, (u_0, b_0)) \leq \overline{W} \text{ for all } \tau > 0,$$

and

$$\underline{b}_\lambda \leq b(\tau, (u_0, b_0)) \leq \overline{b} \text{ for all } \tau > 0,$$

and the same estimates hold when starting from the lower and upper solutions. A next step is to prove that

$$\int_T^{T+1} \int_0^{b(\tau)} W_\eta^2(\eta, \tau)\, d\eta d\tau \leq M_1, \tag{27}$$

where the constant M_1 does not depend on T and where the initial functions for W and b are either lower or upper solutions.

Proof of the gradient bound

We multiply the partial differential equation for W by W and integrate on $\Omega_{T,1} := \{(\eta, \tau) : \eta \in (0, b(\tau)), \tau \in (T, T+1)\}$.This yields

$$L_1(T) = \int_{\Omega_{T,1}} WW_\tau \, d\eta d\tau = \int_{\Omega_{T,1}} (WW_{\eta\eta} + \frac{\eta}{2} W_\eta W) \, d\eta d\tau = R_1(T).$$

Then,

$$2L_1(T) = \int_{\Omega_{T,1}} \frac{\partial}{\partial \tau} W^2(\eta, \tau) \, d\tau d\eta$$

and

$$\int_{\Omega_{T,1}} \frac{\partial}{\partial \tau} W^2(\eta, \tau) \, d\tau d\eta = \int_0^{b(T+1)} W^2(\eta, T+1) \, d\eta - \int_0^{b(T)} W^2(\eta, T) \, d\eta,$$

Proof of the gradient bound

Integration by parts in $R_1(T)$ yields

$$R_1(T) = -\int_{\Omega_{T,1}} W_\eta^2 \, d\tau d\eta + h \int_T^{T+1} W(0,\tau)\, d\eta - \frac{1}{4} \int_{\Omega_{T,1}} W^2 \, d\tau d\eta.$$

Writing that $L_1(T) = R_1(T)$ then yields the result.

A weak form of the Stefan problem

We multiply the partial differential equation by $\varphi \in C_0^\infty(\mathbb{R})$ and integrate by parts. The left-hand-side becomes

$$L_2(T) = \int_T^{T+1} \int_0^{b(\tau)} W_\tau \, \varphi \, d\eta d\tau,$$

$$L_2(T) = \int_0^{b(T+1)} W(\eta, T+1)\varphi(\eta) \, d\eta - \int_0^{b(T)} W(\eta, T)\varphi(\eta) \, d\eta,$$

where $W = W(\eta, \tau, (\bar{W}, \bar{b}))$ or $W = W\big(\eta, \tau, (W_\lambda, b_\lambda)\big)$. Next, we investigate the right-hand-side. Integration by parts yields

$$R_2(T) = \int_T^{T+1} \int_0^{b(\tau)} (W_{\eta\eta} + \frac{\eta}{2} W_\eta)\varphi \, d\eta d\tau,$$

A weak form of the Stefan problem

or else, if $\varphi_\eta(0) = 0$,

$$R_2(T) = \int_T^{T+1} \int_0^{b(\tau)} (W_{\eta\eta}\varphi \, d\eta d\tau + \int_T^{T+1} \int_0^{b(\tau)} \frac{\eta}{2} W_\eta \varphi \, d\eta d\tau,$$

$$= \int_T^{T+1} \int_0^{b(\tau)} W(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta) \, d\eta d\tau - \int_T^{T+1} (\dot{b} + \frac{b}{2})\varphi(b(\tau)) + h\varphi(0). \tag{28}$$

Expressing that $L_2(T) = R_2(T)$ yields the weak form

$$\int_0^{b(T+1)} W(\eta, T+1)\varphi(\eta) \, d\eta - \int_0^{b(T)} W(\eta, T)\varphi(\eta) \, d\eta$$

$$= \int_T^{T+1} \int_0^{b(\tau)} W(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta) \, d\eta d\tau - \int_T^{T+1} (\dot{b} + \frac{b}{2})\varphi(b(\tau)) + h\varphi(0),$$

for all $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi_\eta(0) = 0$.

Limit as $\tau \to \infty$

Next, we pass to the limit as $T \to \infty$. It follows from Lebesgue's dominated convergence theorem that $\lim_{T\to\infty} L_2(T) = 0$. As for the right-hand-side, we obtain

$$\lim_{T\to\infty} \int_T^{T+1} \int_0^{b(\tau)} W(\eta,\tau)(\varphi_{\eta\eta} - (\eta\varphi)_\eta)\, d\eta d\tau$$
$$= \int_0^{b^\infty} W^\infty(\eta)(\varphi_{\eta\eta} - (\eta\varphi)_\eta)\, d\eta,$$

where W^∞ is either ψ or ϕ, and b^∞ is either $\underline{b}^\infty$ or $\overline{b}^\infty$. Let us denote by Φ an antiderivative of φ; then

$$\lim_{T\to\infty} \int_T^{T+1} \dot{b}(\tau)\varphi(b(\tau))d\tau = \lim_{T\to\infty} (\Phi(b(T+1)) - \Phi(b(T))) = 0.$$

In addition,

$$\lim_{T\to\infty} \int_T^{T+1} \frac{b(\tau)}{2}\varphi(b(\tau))d\tau = \frac{1}{2}b^\infty\varphi(b^\infty).$$

Limit as $\tau \to \infty$

Finally, we collect all the results concerning $R_2(T)$, while keeping in mind that $\lim_{T\to\infty} L_2(T) = 0$. This yields

$$\int_0^{b^\infty} W^\infty(\eta)(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta)\, d\eta - \frac{b^\infty}{2}\varphi(b^\infty) + h\varphi(0) = 0, \qquad (29)$$

for all $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi_\eta(0) = 0$. Thus (W^∞, b^∞) is a steady state solution of (17) in the sense of distributions. We now wish to recover the ordinary differential equation as well as the two boundary conditions and the condition on the interface.

Taking $\varphi \in C_0^\infty(0, b^\infty)$, we conclude that W^∞ is a smooth function which satisfies the steady state equation

$$W^\infty_{\eta\eta} = -\frac{\eta}{2} W^\infty_\eta \text{ for all } \eta \in (0, b^\infty).$$

Limit as $\tau \to \infty$

Next we search for the boundary conditions satisfied by W^∞. After integrating (29) twice by parts we obtain,

$$\begin{aligned} 0 =& W^\infty(b^\infty)\varphi_\eta(b^\infty) - W^\infty_\eta(b^\infty)\varphi(b^\infty) + W^\infty_\eta(0)\varphi(0) \\ &- \frac{b^\infty}{2}\varphi(b^\infty)(W^\infty(b^\infty)+1) + h\varphi(0). \end{aligned} \tag{30}$$

Now, if we additionally choose φ such that $\varphi(b^\infty) = \varphi_\eta(b^\infty) = 0$, then (30) reduces to

$$\varphi(0)(W^\infty_\eta(0) + h) = 0,$$

and since $\varphi(0)$ is arbitrary, we deduce that

$$-W^\infty_\eta(0) = h.$$

Limit as $\tau \to \infty$

Thus (30) becomes

$$0 = W^\infty(b^\infty)\varphi_\eta(b^\infty) - W_\eta^\infty(b^\infty)\varphi(b^\infty) - \frac{b^\infty}{2}\varphi(b^\infty)(W^\infty(b^\infty) + 1) \tag{31}$$

Now, suppose that $\varphi(b^\infty) = 0$, but $\varphi_\eta(b^\infty) \neq 0$. Then

$$\varphi_\eta(b^\infty)W^\infty(b^\infty) = 0,$$

and hence

$$W^\infty(b^\infty) = 0.$$

Then (31) becomes

$$0 = -W_\eta^\infty(b^\infty)\varphi(b^\infty) - \frac{b^\infty}{2}\varphi(b^\infty). \tag{32}$$

Suppose that $\varphi(b^\infty) \neq 0$. Then (32) implies that

$$\frac{b^\infty}{2} = -W_\eta^\infty(b^\infty).$$

Limit as $\tau \to \infty$

We deduce that the solution pair (W^∞, b^∞) is the unique steady state solution of the time evolution Problem (17), which coincides up to a change of variables, with the unique self-similar solution of Problem (16).

Our convergence result as $\tau \to \infty$ is essentially the same as the one I have showed you for the Dirichlet problem.

Projects with Professor Yasuhito Miyamoto

Extend our results to the case of
(i) the two-phase Stefan problems with zero latent heat;
(ii) the two-phase Stefan problems with positive latent heat.

Also we propose to consider Stefan problems with nonlinear diffusion in collaboration with Elaine Crooks.

I thank you for your attention