Fractional Nonlinear Diffusion Equation: Numerical Analysis and Large-time Behavior

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Outline

1. Large-time behaviour of fractional nonlinear diffusion equation
   - Derivation of the equation
   - Decay estimates
   - Convergence to asymptotic profiles with energy method

2. Numerical analysis of fractional nonlinear diffusion equation
   - Discretization of the equation
   - Analysis of the numerical scheme
   - Numerical illustrations of the large-time behaviour

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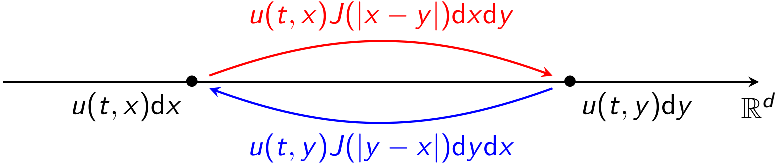
Nonlocal diffusion equation with general interaction kernel



$u(t,x)$: density, $J(|x-y|)$: interaction kernel.

$$\begin{aligned}\partial_t u(t,x) &= \int_{\mathbb{R}^d} u(t,y)J(|x-y|)\mathrm{d}y - u(t,x)\int_{\mathbb{R}^d} J(|x-y|)\mathrm{d}y \\ &= \mathrm{P.V.}\int_{\mathbb{R}^d}(u(t,y)-u(t,x))J(|x-y|)\mathrm{d}y \\ &:= \lim_{r\to 0}\int_{\mathbb{R}^d\setminus B_r(x)}(u(t,y)-u(t,x))J(|x-y|)\mathrm{d}y\end{aligned}$$

for singular kernel J (P.V.)

A particular choice of kernel: the fractional diffusion equation

Choice of kernel: $J_{\frac{\alpha}{2}}(|x-y|) = \frac{1}{|x-y|^{d+\alpha}}, \quad \alpha \in (0,2).$

Definition (Fractional Laplacian)

$$(-\Delta)^{\frac{\alpha}{2}} u(x) := C_{d,\alpha} \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+\alpha}} \mathrm{d}y,$$

$$C_{d,\alpha} := \frac{\alpha 2^{\alpha-1} \Gamma\left(\frac{\alpha+d}{2}\right)}{\pi^{d/2} \Gamma\left(\frac{2-\alpha}{2}\right)}.$$

- $\mathcal{F}[(-\Delta)^{\frac{\alpha}{2}} u](\xi) = |\xi|^{\alpha} \mathcal{F}[u](\xi)$,
- $J_{\frac{\alpha}{2}}$ has a non-integrable singularity, and is heavy-tailed.

Fractional diffusion equation: $\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u = 0.$

Fractional nonlinear diffusion equation on bounded domain

$$\partial_t u + (-\Delta)^{\frac{\alpha}{2}} u^m = 0.$$

- $m > 1$: *fractional porous medium equation*,
- $0 < m < 1$: *fractional fast diffusion equation*.

Change of variable: $q := 1/m + 1$, $v = u^m \;\Rightarrow\; \partial_t(v^{q-1}) + (-\Delta)^{\frac{\alpha}{2}} v = 0.$

Fractional nonlinear diffusion equation on a bounded domain Ω

$$\begin{cases} \partial_t v^{q-1} + (-\Delta)^{\frac{\alpha}{2}} v = 0 & \text{in } \Omega \times (0, +\infty), \\ v = 0 & \text{in } (\mathbb{R}^d \setminus \Omega) \times (0, +\infty), \\ v(\cdot, 0) = v_0 & \text{in } \Omega. \end{cases} \tag{CDP}$$

Purpose

$$\partial_t(v^{q-1}) + (-\Delta)^{\frac{\alpha}{2}} v = 0.$$

- $q \in (1,2)$: *fractional porous medium equation*,
- $q \in (2,+\infty)$: *fractional fast diffusion equation*.

Purpose:

- Energy decay estimates,
- Fast diffusion case: Near extinction asymptotics,
- Numerical scheme preserving energy decay estimates.

Energy decay estimates: porous medium case

Proposition ([Bonforte, Vazquez], [Akagi, S.])

Assume $q \in (1,2)$. *Let* v *be an energy solution of* (CDP).
There exist $c, C > 0$ *such that, for any* $t > 0$,

$$\left(\|v^0\|_{L^q(\Omega)}^{q-2} + ct\right)^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\Omega)} \leq \left(\|v^0\|_{L^q(\Omega)}^{q-2} + Ct\right)^{\frac{1}{q-2}}.$$

Energy decay estimates: fast diffusion case

Proposition ([Bonforte, Ibarrondo, Ispizua], [Akagi, S.])

Assume $q \in (2, 2^*_\alpha]$. *Let* v *be an energy solution of* (CDP).
There exist $c, C > 0$ *such that, for any* $t > 0$,

$$\left(\|v_0\|^{q-2}_{L^q(\Omega)} - ct\right)_+^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\Omega)} \leq \left(\|v_0\|^{q-2}_{L^q(\Omega)} - Ct\right)_+^{\frac{1}{q-2}}.$$

In particular, u *extincts at a time* $t_* \leq T_* := \frac{\|v^0\|^{q-2}_{L^q(\mathbb{R}^d)}}{C}$.

Moreover, for any $t > 0$,

$$c(t_* - t)_+^{\frac{1}{q-2}} \leq \|v(t)\|_{L^q(\Omega)} \leq C(t_* - t)_+^{\frac{1}{q-2}},$$

and the same is true when $\|\cdot\|_{L^q(\Omega)}$ is replaced by $\|\cdot\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)}$.

Idea of proof

Lemma (Fractional Sobolev inequality)

Let $2^*_\alpha := \frac{2d}{(d-\alpha)_+}$, *and* $q \in (1, 2^*_\alpha]$. *There exists* $K > 0$ *such that,*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq K[u]_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} = K\sqrt{\left((-\Delta)^{\frac{\alpha}{2}}u, u\right)_{L^2(\mathbb{R}^d)}},$$

for any $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ *with* $u \equiv 0$ *in* $\mathbb{R}^d \setminus \Omega$.

1. Obtain *energy identity* from the variational form of the equation,
2. Use the fractional Sobolev inequality, or monotonicity of Rayleigh quotient, to obtain an ordinary differential inequality,
3. Integrate the ordinary differential inequality.

Assume $2 < q < 2^*_\alpha$. Then

$$c(t_* - t)_+^{\frac{1}{q-2}} \leq \|v(t)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} \leq C(t_* - t)_+^{\frac{1}{q-2}}.$$

Rescaled solution: $w(s) := (t_* - t)^{\frac{-1}{q-2}} v(t), \quad s := \log\left(\frac{t_*}{t_* - t}\right).$

Then $\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{q-2} w^{q-1}$, and $c < \|w(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < C$

Does $w(s)$ converges as $s \to \infty$?

Non-fractional case:

- Berryman, Holland ('80): convergence along subsequences,
- Feireisl, Simondon ('00): convergence along the full sequence,
- Bonforte, Grillo, Vázquez ('12): convergence in relative error,
- Bonforte, Figalli ('20), Jin, Xiong ('23), Akagi ('23), Choi, McCann, Seis ('23): Sharp rate of convergence.

Rescaled solution and asymptotic profiles

$$\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{q-2} w^{q-1}, \text{ and } c < \|w(s)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < C.$$

Proposition ([Akagi, S.])

For any $s_n \to +\infty$, there exists a subsequence (still denoted by (s_n)), and $\phi \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) \setminus \{0\}$, with $\phi = 0$ in $\mathbb{R}^d \setminus \Omega$, such that

$$w(s_n) \to \phi \quad \text{strongly in } H^{\frac{\alpha}{2}}(\mathbb{R}^d),$$
$$(-\Delta)^{\frac{\alpha}{2}} \phi = \lambda_q \phi^{q-1} \quad \text{in } \Omega,$$

with $\lambda_q := \frac{q-1}{q-2} > 0$.

Generalized gradient flow structure for nonlinear diffusion

Define

$$\mathcal{X}_{\frac{\alpha}{2}}(\Omega) := \{u \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) : \; u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\},$$
$$J(w) := \frac{1}{2}\|w\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} - \frac{\lambda_q}{q}\|w\|^q_{L^q(\Omega)} \quad \text{for} \;\; w \in \mathcal{X}_\theta(\Omega).$$

Then the rescaled solution $s > 0 \mapsto w(s) \in \mathcal{X}_\theta(\Omega)$ solves

$$\partial_s w^{q-1}(s) = -J'(w(s)), \quad \text{for a.e. } s > 0.$$

Therefore it holds

$$\frac{4}{qq'}\left\|\partial_s(|w|^{(q-2)/2}w)(s)\right\|^2_{L^2(\Omega)} + \frac{\mathrm{d}}{\mathrm{d}s}J(w(s)) \le 0, \quad \text{for a.e. } s > 0,$$

and $J(w(\cdot))$ is non-increasing.

Łojasiewicz inequality

Theorem ([Łojasiewicz])

Let $U \subset \mathbb{R}^d$ open and $f : U \to \mathbb{R}$ a real-analytic function. Let $x_0 \in U$ such that $\nabla f(x_0) = 0$. Then there exists a neighborhood V of x_0, $\omega > 0$ and $\theta \in (0, 1/2]$ such that

$$|f(x) - f(x_0)|^{1-\theta} \leq \omega |\nabla f(x)|, \quad x \in V.$$

Proof for $d = 1$: $\exists N \geq 2$ s.t. $f^{(N)}(x_0) \neq 0$ and

$$f(x_0 + h) - f(x_0) = \frac{f^{(N)}(x_0)}{N!} h^N + o(h^N),$$

$$f'(x_0 + h) = \frac{f^{(N)}(x_0)}{(N-1)!} h^{N-1} + o(h^{N-1})$$

$$\Rightarrow (f(x_0 + h) - f(x_0))^{\frac{N-1}{N}} = \frac{1}{N f^{(N)}(x_0)^{1/N}} f'(x_0 + h) + \underbrace{o(h^{N-1})}_{\lesssim f'(x_0+h)}$$

Łojasiewicz inequality implies full convergence of gradient flows

Lemma ([Łojasiewicz])

Let $x \in \mathrm{AC}_{\mathrm{loc}}(0,\infty;\mathbb{R}^{\mathrm{d}})$ be a solution to $\dot{x}(t) = -\nabla f(x(t))$ for a.e. $t > 0$. Assume $\exists\ (t_n)_n$ s.t. $t_n \nearrow \infty$ and $x(t_n) \to x_\infty$ with $\nabla f(x_\infty) = 0$. Then $x(t) \to x_\infty$ as $t \to +\infty$.

Sketch of proof:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f(x(t)) - f(x_\infty)) = -|\nabla f(x(t))||\dot{x}(t)| \leq -\omega(f(x(t)) - f(x_\infty))^{1-\theta}|\dot{x}(t)|$$

$$\Rightarrow |\dot{x}(t)| \leq -\frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}(f(x(t)) - f(x_\infty))^{\theta}$$

$$\Rightarrow \int_0^\infty |\dot{x}(t)|\mathrm{d}t < \frac{1}{\omega}(f(x(0)) - f(x_\infty))^{\theta} < \infty$$

PDE case: Łojasiewicz-Simon inequality

Simon ('83) extended Łojasiewicz inequality to certain analytic functionals. Applicable to semilinear parabolic equation with analytic nonlinearities.

Feiresl and Simondon ('00) extended Łojasiewicz-Simon inequality to the standard Laplacian and non-analytic nonlinearities.

Akagi, Schimperna, Segatti ('19) extended Łojasiewicz-Simon inequality to the fractional Laplacian and to non-analytic nonlinearities.

Łojasiewicz-Simon inequality for fractional Laplacian

$$I(w) := \frac{1}{2}\|w\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} + \int_\Omega \int_0^{w(x)} g(s)\mathrm{d}s\mathrm{d}x \quad \text{for } w \in H^{\frac{\alpha}{2}}(\mathbb{R}^d),\ \hat{g}(w(\cdot)) \in L^1(\Omega).$$

(H0) $g \in C^1(\mathbb{R})$ and $g(0) = 0$,

(H1) $g \in C^\infty((0,\infty))$, and for all $\beta \in (0,\infty)$, there exist $C, M \geq 0$ such that,

$$|g^{(n)}(s)| \leq C\frac{M^n n!}{|s|^n}, \quad \forall s \in (0,\beta),\ n \in \mathbb{N}$$

(H2) there exists $0 \leq p < \infty$ with $p \leq 2^*_\alpha - 1$ such that

$$|g'(s)| \leq C(|s|^{p-1} + 1) \quad \text{for all } s \in \mathbb{R}.$$

Lemma ([Akagi, Schimperna, Segatti])

Assume (H0), (H1), (H2), and let $\psi \in H^{\frac{\alpha}{2}}(\mathbb{R}^d) \cap L^\infty(\Omega)$ such that $I'(\psi) = 0$ and $\psi > 0$. Then, there exists $\theta \in (0, 1/2]$ and $\omega, \delta > 0$ s.t.

$$|I(w) - I(\psi)|^{1-\theta} \leq \omega \|I'(w)\|_{H^{-\frac{\alpha}{2}}(\Omega)}, \quad \text{if } \|w-\psi\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} < \delta.$$

Full convergence to the asymptotic profile

Proposition ([Akagi, S.])

Let $2 < q < 2^*_\alpha$. Assume that $\phi > 0$ and $w(s_n) \to \phi$ strongly in $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ for some sequence of times $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to +\infty$. Then

$$w(s) \to \phi \quad \text{strongly in } H^{\frac{\alpha}{2}}(\mathbb{R}^d) \text{ as } s \to +\infty.$$

Idea of proof: Use the Łojasiewicz-Simon inequality with the functionnal

$$J(w) := \frac{1}{2}\|w\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^d)} - \frac{\lambda_q}{q}\|w\|^q_{L^q(\Omega)}.$$

Open problems:

- Estimation of the extinction time,
- Rate of convergence to the asymptotic profile.

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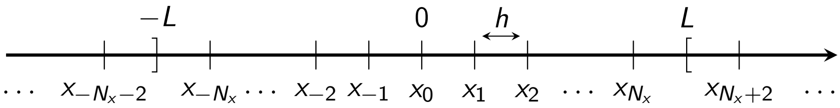
Notation

We restrict to dimension $d = 1$.

- $\Omega = (-L, L)$, space step $h = L/(N_x + 1)$:



- time step $\Delta t > 0$.

For u a fonction over $[0, +\infty) \times \mathbb{R}$,

$$u_i^n := u(n\Delta t, ih),$$

$$\|u^n\|^q_{l^q_h(\mathbb{R})} := \sum_{i\in\mathbb{Z}} |u_i|^q h.$$

Discretization of the fractional Laplacian

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) := C_{d,\alpha}\left(\underbrace{\text{P.V.}\int_{|x_i-y|<h} \frac{u_i - u(y)}{|x_i - y|^{d+\alpha}}\mathrm{d}y}_{\text{singular part}} + \underbrace{\int_{|x_i-y|>h} \frac{u_i - u(y)}{|x_i - y|^{d+\alpha}}\mathrm{d}y}_{\text{tail part}} \right)$$

- singular part: $u(y)$ replaced by Taylor expansion,
- tail part: $u(y)$ replaced by piecewise quadratic interpolation.

Then integrating *explicitly* yields, for some weights $(\gamma_j^h)_{j\in\mathbb{Z}}$,

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) \approx \sum_{j\in\mathbb{Z}} \gamma_j^h (u_i - u_{i-j}).$$

Convergence result: For $u \in C^4$, the error is in $\mathcal{O}(h^{3-\alpha})$.

➢ Y. Huang and A. Oberman. "Numerical Methods for the Fractional Laplacian: A Finite Difference-Quadrature Approach". In: (2014)

Convolution structure of the discrete fractional Laplacian

$$\left[(-\Delta)_h^{\frac{\alpha}{2}} u\right]_i := \sum_{j\in\mathbb{Z}} \gamma_j^h (u_i - u_{i-j})$$

$$\updownarrow$$

$$(-\Delta)^{\frac{\alpha}{2}} u(x_i) = \text{P.V.} \int_{\mathbb{R}^d} \frac{C_{1,\alpha}}{|z|^{1+\alpha}} (u(x_i) - u(x_i - z))\mathrm{d}z.$$

Theorem ([Ayi, Herda, Hivert, Tristani, 2022])

There exists positive constants b_α and B_α independent of h such that

$$\frac{b_\alpha}{|jh|^{1+\alpha}} h \leq \gamma_j^h \leq \frac{B_\alpha}{|jh|^{1+\alpha}} h.$$

Convolution structure of the discrete fractional Laplacian

For u a Schwartz function,

$$\sum_{i\in\mathbb{Z}}\left[(-\Delta)_h^{\frac{\alpha}{2}}u\right]_i u_i = \frac{1}{2}\underbrace{\sum_{i\in\mathbb{Z},j\in\mathbb{Z}} h\gamma_j^h|u_i-u_{i-j}|^2}_{=:[u]^2_{H_h^{\frac{\alpha}{2}}(\mathbb{R})}} \sim [u]^2_{H^{\frac{\alpha}{2}}(\mathbb{R})},$$

Lemma ([Hivert, S.], Discrete fractional Sobolev inequality)

For $q\leq 2^*_\alpha$, there exists $K>0$ independent of h,

$$\|u\|_{l_h^q(\mathbb{R})}\leq K[u]_{H_h^{\frac{\alpha}{2}}(\mathbb{R})},$$

for $u\in\mathbb{Z}^{\mathbb{N}}$ with $u\equiv 0$ outside Ω.

Numerical scheme for fractional nonlinear diffusion equation

Implicit scheme for CDP:

$$\begin{cases} \dfrac{(u_i^{n+1})^{q-1} - (u_i^{n})^{q-1}}{\Delta t} + \left[(-\Delta)_h^{\frac{\alpha}{2}} u^{n+1}\right]_i = 0, & |i| \leq N_x \text{ and } n \geq 0, \\ u_i^n = 0, & |i| \geq N_x + 1 \text{ and } n \geq 0, \\ u_i^0 = (u^0)_i & |i| \leq N_x. \end{cases}$$

Does the scheme has the same property as the continuous equation (decay rate, extinction) ?

Discrete decay estimates: porous medium case

Proposition ([Hivert, S.])

Assume $q \in (1,2)$. *There exists* $(\beta_n^{\Delta t})_{n \geq 0}$, *independent of* h, *such that*

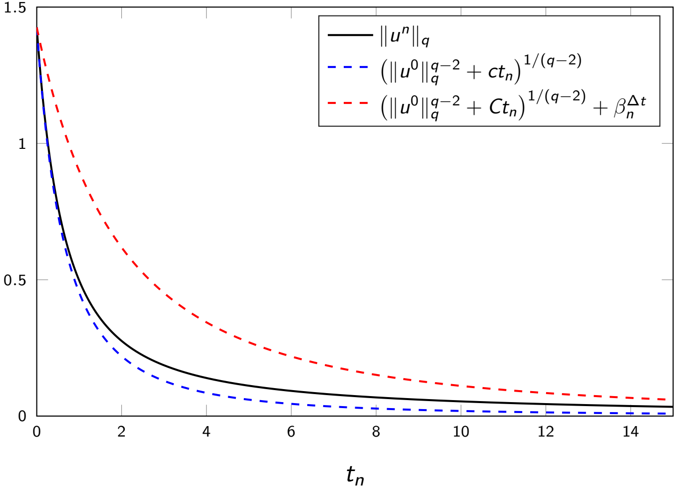
$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \left(\|u^0\|_{l_h^q(\mathbb{R})}^{q-2} + Cn\Delta t\right)^{\frac{-1}{2-q}} + \beta_n^{\Delta t}, \quad \text{for any } n \geq 0.$$

Moreover,

$$\sup_{n \geq 0} \beta_n^{\Delta t} \to 0, \quad \text{as } \Delta t \to 0.$$

Numerical results: energy decay for PME

Figure: Energy decay for $q = 1.5, \alpha = 0.5, h = 0.04, \Delta t = 0.03,\ L = 5$.



Discrete decay estimates: fast diffusion case

Proposition ([Hivert, S.])

Assume $q \in (2, 2^*_\alpha]$.

- *Decay estimate: There exists* $(\beta_n^{\Delta t})_{n \geq 0}$, *independent of* h, *such that*

$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \left(\|u^0\|_{l_h^q(\mathbb{R})}^{q-2} - Cn\Delta t\right)_+^{\frac{1}{q-2}} + \beta_n^{\Delta t}, \quad \text{for any } n \geq 0.$$

  Moreover,

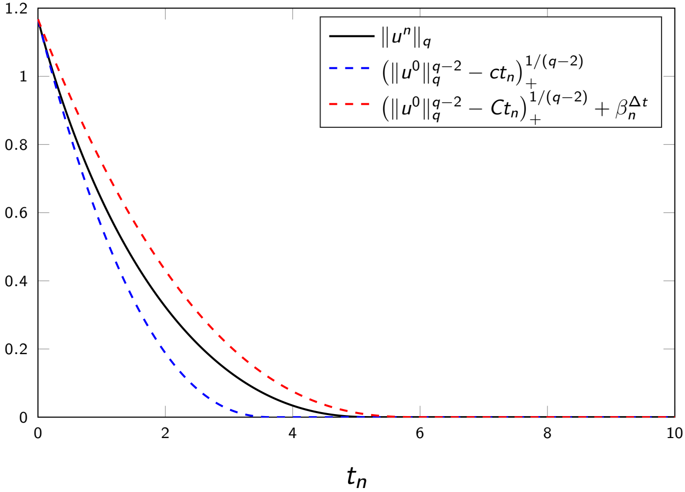
$$\sup_{n \geq 0} \beta_n^{\Delta t} \to 0, \quad \text{as } \Delta t \to 0.$$

- *Extinction estimate:*

$$\|u^n\|_{l_h^q(\mathbb{R})} \leq \|u^0\|_{l_h^q(\mathbb{R})} \left(\frac{T_*}{n\Delta t}\right)^{n/2}, \quad \text{for any } n \geq 0.$$

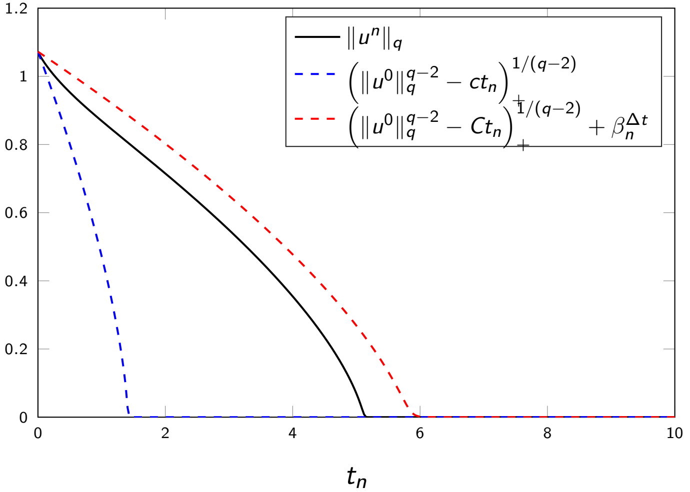
Numerical results: energy decay for FDE

Figure: Energy decay for $q = 2.4, \alpha = 0.5, h = 0.04, \Delta t = 0.03,\ L = 5.$



Numerical results: energy decay for FDE

Figure: Energy decay for $q = 3.5, \alpha = 1.5, h = 0.04, \Delta t = 0.03,\ L = 5.$



Idea of proof

1. Obtain energy *inequality*,

$$\frac{1}{q'}\frac{\|u^{n+1}\|^q_{l^q_h(\mathbb{R})} - \|u^n\|^q_{l^q_h(\mathbb{R})}}{\Delta t} + \|u^{n+1}\|^2_{H_h^{\frac{\alpha}{2}}(\mathbb{R})} \leq 0.$$

2. Use *discrete* fractional Sobolev inequality to obtain a discretization of the ordinary differential inequality,

$$\frac{1}{q'}\frac{\|u^{n+1}\|^q_{l^q_h(\mathbb{R})} - \|u^n\|^q_{l^q_h(\mathbb{R})}}{\Delta t} + K^{-2}\|u^{n+1}\|^2_{l^q_h(\mathbb{R})} \leq 0.$$

3. Sum in time and use convexity inequalities.

Computation of the extinction time when $q > 2$

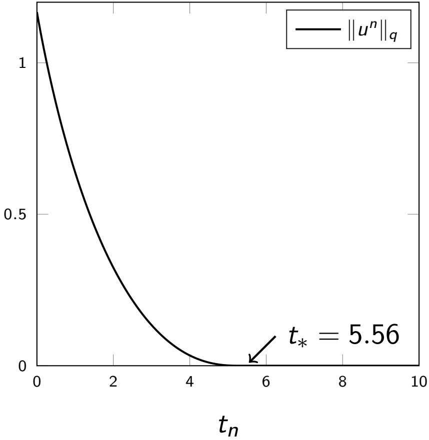
u : solution to the continuous problem with initial data u^0
$t_*(u^0)$: extinction time of u
Let $\tilde{t} > 0$ and $w(s) := (\tilde{t} - t)^{\frac{-1}{q-2}} u(t)$, $s := \log(\tilde{t}/(\tilde{t} - t))$. Then

- $\partial_s w^{q-1} + (-\Delta)^{\frac{\alpha}{2}} w = \frac{q-1}{q-2} w^{q-1}, \quad w(0) = \tilde{t}^{\frac{-1}{q-2}} u(0),$
- $\|w(s)\|_q \to \infty$ as $s \to \infty$ if $\tilde{t} > t_*(u^0)$,
- w extinct in finite time if $\tilde{t} < t_*(u^0)$.

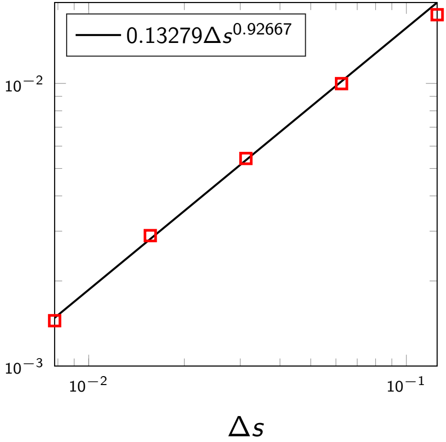
To compute $t_*(u^0)$ we proceed by dichotomy using the scheme

$$\begin{cases} \dfrac{(w_i^{n+1})^{q-1} - (w_i^n)^{q-1}}{\Delta s} + \left[(-\Delta)_h^{\frac{\alpha}{2}} w^{n+1}\right]_i = \dfrac{q-1}{q-2}(w^{n+1})^{q-1}, \\ \hfill w_i^0 = \tilde{t}^{-1/(q-2)}(u^0)_i. \end{cases}$$

Figure: $q = 2.4, \alpha = 0.5, h = 0.04, \Delta t = 0.03,\ L = 5$
t_* computed by dichotomy with $\Delta s = 0.001$



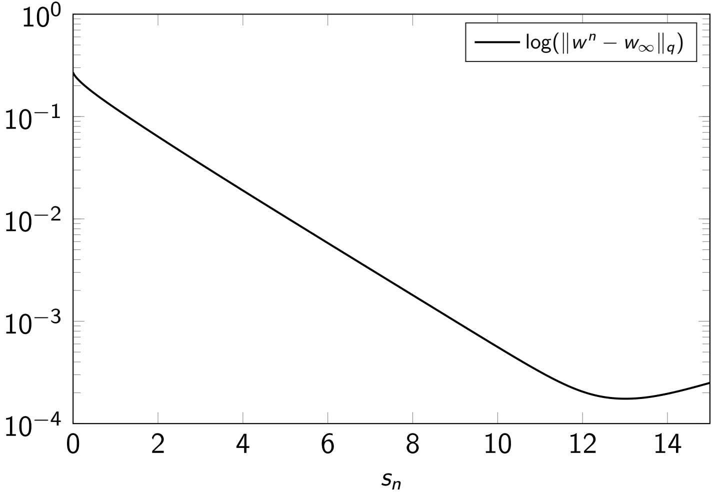
(a) Energy decay of the non-rescaled solution



(b) Convergence of the computation of the extinction time as $\Delta s \to 0$

Figure: $w(t)$ for $\alpha = 0.5$, $q = 2.4 < 2^*_\alpha$, $h = 0.01$, $\Delta s = 0.01$
t_* computed by dichotomy with $\Delta s = 0.01$
w^∞: theoretical asymptotic profile

Figure: $q = 2.4 < 2^*_\alpha$, $h = 0.01$, $\Delta s = 0.01$
t_* computed by dichotomy with $\Delta s = 0.01$
w^∞: theoretical asymptotic profile



Summary

We showed convergence to asymptotic profiles in fast diffusion case.
We introduced a numerical scheme having same decay energy as the fractional nonlinear diffusion equation, and a method for the computation of the extinction time.

Extensions:

- Rates of convergence to asymptotic profiles,
- Numerical analysis in dimension $d \geq 1$,
- Better convergence results for the numerical scheme.

Thank you for your attention!